

Edge- and Node-Disjoint Paths in P Systems

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In this paper, we continue our development of algorithms used for topological network discovery. We present native P system versions of two fundamental problems in graph theory: finding the maximum number of edge- and node-disjoint paths between a source node and target node. We start from the standard depth-first-search maximum flow algorithms, but our approach is totally distributed, when initially no structural information is available and each P system cell has to even learn its immediate neighbors. For the node-disjoint version, our P system rules are designed to enforce node weight capacities (of one), in addition to edge capacities (of one), which are not readily available in the standard network flow algorithms.

Keywords: P systems, P modules, simple P modules, cell IDs, distributed algorithms, synchronous networks, breadth-first-search, depth-first-search, edge-disjoint paths, node-disjoint paths, network flow, network discovery, routing.

1 Introduction

Inspired by the structure and interaction of living cells, P systems provides a distributed computational model, as introduced by G. Păun in 1998 [12]. The model was initially based on transition rules, but was later expanded into a large family of related models, such as tissue and neural P systems (nP systems) [7, 13] and hyperdag P systems (hP systems) [10]. Essentially, all versions of P systems have a structure consisting of cell-like membranes and a set of rules that govern their evolution over time. A large variety of rules have been used to describe the operational behavior of P systems, the main ones being: multiset rewriting rules, communication rules and membrane handling rules. Transition P systems and nP systems use multiset rewriting rules, P systems with symport/antiport operate by communicating immutable objects, P systems with active membranes combine all three type rules. For a comprehensive overview and more details, we refer the reader to [13].

Earlier in [3], we have proposed an extensible framework called P modules, to assist the programmability of P systems. P modules enable the modular composition of complex P systems and also embrace the essential features of a variety of P systems. In this paper, we will use a restricted subset of this unifying model, called simple P modules, (subset equivalent to neural P systems [7]), to develop algorithms for finding the maximum number of edge- and node-disjoint paths between two cells in a fairly large class of P systems, where duplex communication channels exist between neighboring cells. We assume that the digraph structure of the simple P module is completely unknown (even the local neighboring cells must be discovered [9]) and we need to, via a distributed process, optimally create local routing tables between a given source and target cell.

There are endless natural applications that need to find alternative routes between two points, from learning strategies to neural or vascular remodeling after a stroke. In this paper, we focus on a related but highly idealized goal, how to compute a maximum cardinality set of edge- and node-disjoint paths between two arbitrary nodes in a given digraph.

One obvious application related to networks is to find the best bandwidth utilization for routing of information between a source and target [15]. For instance, streaming of applications for multi-core computations uses edge-disjoint paths routing for task decomposition and inter-task communications [17]. In fact, classical solutions are based on a network flow approach such as given in [5, 4], or on Menger's Theorem, an old, but very useful, result, cited below.

Theorem 1 (Menger [8]). *Let $D = (V, A)$ be a digraph and let $s, t \in V$. Then the maximum number of node-disjoint $s-t$ paths is equal to the minimum size of an $s-t$ disconnecting node set.*

Another application is to find a maximum matching (or pairing) between two compatible sets such as the marriage arrangement problem or assigning workers to jobs.

Our third application (and a motivating problem for the authors) is the Byzantine Agreement problem [3, 2], in the case of non-complete graphs. The standard solution (also based on Menger's Theorem) allows for k faulty nodes (within a set of nodes of order at least $3k + 1$) if and only if there are at least $2k + 1$ node-disjoint paths between each pair of nodes, to ensure that a distributed consensus can occur [6].

Briefly, the paper is organized as follows. In the next section, we give a formal definition of simple P modules, to give a unified platform for developing our P systems algorithms. Next, in Section 3 we summarize the standard network flow approaches for finding edge- and node-disjoint paths in digraphs and we discuss optimizations and alternative strategies which are more appropriate for P systems. In Section 4, we discuss three possible relations between the structural digraph underlying a simple P module and the search digraph used for determining paths. Section 5 details breadth-first-search rules used to determine the local cell topologies, i.e. all cell neighborhoods; this is a common preliminary phase for both the edge- and node-disjoint path implementations. The next two sections detail depth-first-search rules for the edge-disjoint case (in Section 6) and for the node-disjoint case (in Section 7). Finally, in Section 8, we end with conclusions and some open problems.

2 Preliminary

We assume that the reader is familiar with the basic terminology and notations: functions, relations, graphs, edges, nodes (vertices), directed graphs, arcs, paths, directed acyclic graphs (dags), trees, alphabets, strings and multisets [11]. We now introduce *simple P modules*, as a unified model for representing several types of P systems. Simple P modules are a simplified variety of the full P modules, which omit the extensibility features and use duplex communication channels only. With these restrictions, although their formal definitions are different, simple P modules are essentially equivalent to neural P systems [7]. For the full definition of P modules and further details on recursive modular compositions, the reader is referred to [3].

Definition 2 (simple P module). A *simple P module* is a system $\Pi = (O, K, \delta)$, where:

1. O is a finite non-empty alphabet of *objects*;
2. $K = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is a finite set of (internal) *cells*;
3. δ is a binary relation on K , without reflexive or symmetric arcs, which represents a set of parent-child structural arcs between existing cells, with *duplex* communication capability.

Each cell, $\sigma_i \in K$, has the initial form $\sigma_i = (Q_i, s_0, w_0, R_i)$ and general form $\sigma_i = (Q_i, s, w, R_i)$, where:

- Q_i is a finite set of *states*;

- $s_0 \in Q_i$ is the *initial state*; $s \in Q_i$ is the *current state*;
- $w_0 \in O^*$ is the *initial multiset* of objects; $w \in O^*$ is the *current multiset* of objects;
- R_i is a finite *ordered* set of multiset rewriting *rules* of the general form: $s x \rightarrow_{\alpha} s' x' (u)_{\beta\gamma}$, where $s, s' \in Q, x, x' \in O^*, u \in O^*, \alpha \in \{\text{min}, \text{max}\}, \beta \in \{\uparrow, \downarrow, \updownarrow\}, \gamma \in \{\text{one}, \text{spread}, \text{rep1}\} \cup K$. If $u = \lambda$, denoting the *empty string* of objects, this rule can be abbreviated as $s x \rightarrow_{\alpha} s' x'$. The application of a rule takes two sub-steps, after which the cell's current state s and multi-set of objects x is replaced by s' and x' , respectively, while u is a message which is sent as specified by the transfer operator $\beta\gamma$.

The rules given by the ordered set(s) R_i are applied in the *weak priority* order [14]. For a cell $\sigma_i = (Q_i, t, w, R_i)$, a rule $s x \rightarrow_{\alpha} s' x' (u)_{\beta\gamma} \in R_i$ is *applicable* if $t = s$ and $x \subseteq w$. Additionally, if $s x \rightarrow_{\alpha} s' x' (u)_{\beta\gamma}$ is the first applicable rule, then each subsequent applicable rule's target state (i.e. state indicated in the right-hand side) must be s' . The semantics of the rules and the meaning of operators α, β, γ are now described.

For convenience, we will often identify a cell σ_i with its index (or *cell ID*) i , when the context of the variable i is clear. We accept that cell IDs appear as objects or indices of complex objects. Also, we accept *custom cell ID* rules, which distinguish the cell ID of the current cell from other cell IDs. For example, the rule 0.1 of Section 5, given as “ $s_0 g_i \rightarrow_{\text{min}} s_0$ ” for cell σ_i , appears as “ $s_0 g_1 \rightarrow_{\text{min}} s_0$ ” in cell σ_1 and as “ $s_0 g_2 \rightarrow_{\text{min}} s_0$ ” in cell σ_2 .

The rewriting operator $\alpha = \text{max}$ indicates that an applicable rewriting rule of R_i is applied as many times as possible, while the operator $\alpha = \text{min}$ requires a rule of R_i is applied only once. The communication structure is based on the underlying digraph structure. In this paper, and we will only use the $\beta = \updownarrow$ and $\gamma \in \{\text{rep1}\} \cup K$ transfer operators. With reference to cell σ_i , a rewriting rule using $(u)_{\updownarrow_{\text{rep1}}}$ indicates that the multiset u is replicated and sent to all neighboring cells (parents and children), i.e. to all cells in $\delta(i) \cup \delta^{-1}(i)$. Assuming that cell σ_j is a parent or a child of cell σ_i , i.e. $j \in \delta(i) \cup \delta^{-1}(i)$, a rewriting rule using $(u)_{\uparrow_j}$ indicates that the multiset u is specifically sent cell σ_j . Otherwise, if $j \notin \delta(i) \cup \delta^{-1}(i)$, the rule is still applied, but the message u is silently discarded. The other non-deterministic transfer operators (e.g., *one*, *spread*, \uparrow , \downarrow) are just mentioned here for completeness, without details, and are not used in this paper. For details, the interested reader is referred to [3].

Remark 3. This definition of simple P module subsumes several earlier definitions of P systems, hP systems and nP systems. If δ is a *tree*, then Π is essentially a tree-based P system (which can also be interpreted as a cell-like P system). If δ is a *dag*, then Π is essentially an hP system. If δ is a *digraph*, then Π is essentially an nP system.

3 Disjoint paths in digraphs

We now briefly describe the basic edge- and node-disjoint paths algorithms, based on network flow, particularized for unweighted edges (i.e. all edge capacities are one), see Ford and Fulkerson [5]. Our presentation will largely follow the standard approach, but also propose a couple of customizations and optimizations, specifically targeted for running on highly distributed and parallel computing models, such as P systems.

We are given a digraph $G = (V, E)$ and two nodes, a source node, $s \in V$, and a target node, $t \in V$. We consider the following two optimization problems: (1) find a maximum cardinality set of edge-disjoint paths from s to t ; and (2) find a maximum cardinality set of node-disjoint paths from s to t . Obviously, any set of node-disjoint paths is also edge-disjoint, but the converse is not true. For example:

- Figure 1 (a) shows a maximum cardinality set of edge-disjoint paths for a digraph G , which is also a maximum cardinality set of node-disjoint paths,
- Figure 1 (b) shows two maximum cardinality sets of edge-disjoint paths for the same digraph G , which are not node-disjoint.
- Figure 2 shows a digraph where the maximum number of edge-disjoint paths is greater than the maximum number of node-disjoint paths.

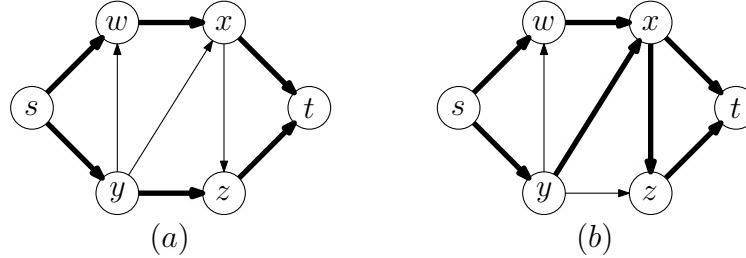


Figure 1: For this digraph, the maximum number of edge-disjoint paths from s to t , which is 2, can be achieved in three ways: (a) paths set $\{s.w.x.t, s.y.z.t\}$; (b) either of the following two paths sets: $\{s.w.x.t, s.y.x.z.t\}$, $\{s.w.x.z.t, s.y.x.t\}$. Paths shown in (a) are also node-disjoint, but paths shown in (b) are not.

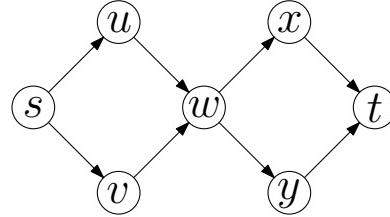


Figure 2: For this digraph, the maximum number of edge-disjoint paths from s to t (2) is greater than the maximum number of node-disjoint paths (1).

3.1 Edge disjoint paths in digraphs

In both edge- and node-disjoint cases, the basic algorithms work by repeatedly searching paths, called augmenting paths, in an auxiliary structure, called residual network or residual digraph. We will first focus more on *edge-disjoint paths*, because the *node-disjoint paths* can be considered as an edge-disjoint paths problem, with additional constraints.

For the following “network flow” definition for digraphs with non-weighted arcs, we say that an arc (u, v) is in a set of paths P , denoted by the slightly abused notation $(u, v) \in P$, if there exists a path $\pi \in P$ that uses arc (u, v) .

Definition 4. Consider a digraph $G = (V, E)$, two nodes s and t , $\{s, t\} \subseteq V$, and a set P of edge-disjoint paths from s to t . Nodes in P are called *flow-nodes* and arcs in P are called *flow-arcs*.

Given path $\pi \in P$, each flow-arc $(u, v) \in \pi$ has a natural *incoming* and *outgoing* direction—the flow is from the source to the target; with respect to π , u is the *flow-predecessor* of v and v is the *flow-successor* of u .

The *residual digraph* is the digraph $R = (V, E')$, where the arcs in P are reversed, or, more formally, $E' = (E \setminus \{(u, v) \mid (u, v) \in P\}) \cup \{(v, u) \mid (u, v) \in P\}$. Any path from s to t in R is called an *augmenting path*.

Given augmenting path α , each flow-arc $(u, v) \in \alpha$ has also a natural *incoming* and *outgoing* direction—the flow is from the source to the target; with respect to α , u is the *search-predecessor* of v and v is the *search-successor* of u .

Fact 5. Augmenting paths can be used to construct a larger set of edge-disjoint paths. More precisely, consider a digraph G and two nodes s and t . A set P_k of k edge-disjoint paths from s to t and an augmenting path α from s to t can be used together to construct a set P_{k+1} of $k+1$ edge-disjoint paths. First, paths in $\{\alpha\} \cup P_k$ are fragmented, by removing “conflicting” arcs, i.e. arcs that appear in $Q \cup \tilde{Q}$, where $Q = P \cap \tilde{\alpha}$ (where \sim indicates arc reversal). Then, new paths are created by concatenating resulting fragments. For the formal definition of this construction, we refer the reader to Ford and Fulkerson [5]. Note that including a reversed arc in an augmenting path is known as *flow pushback operation*.

This construction is illustrated in Figure 3. Figure 3 (a) illustrates a digraph G and a set P_1 of edge-disjoint paths from s to t , currently the singleton $\{\pi_0\}$, where $\pi_0 = s.y.x.t$. Figure 3 (b) shows its associated residual digraph R (note the arcs reversal). Figure 3 (c) shows an augmenting path α in R , $\alpha = s.w.x.y.z.t$. Figure 3 (d) shows the extended set P_2 (after removing arcs (x, y) and (y, x)), consisting of two edge-disjoint paths from s to t , $\pi_1 = s.w.x.t$, $\pi_2 = s.y.z.t$. Figure 4 shows a similar scenario, where another augmenting path is found. Note that the two paths illustrated in Figure 3 (d) form both a maximum edge-disjoint set and a maximum node-disjoint set; however, the two paths sets shown in Figure 4 (d) form two other maximum edge-disjoint path sets, but none of them is node-disjoint.

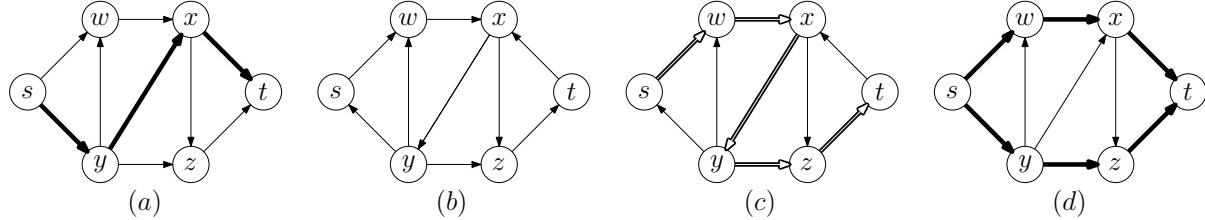


Figure 3: A residual digraph and an augmenting path: (a) a digraph G and one (edge-disjoint) path π_0 from s to t (indicated by bold arrows). (b) the residual digraph R_0 associated to digraph G and path π_0 . (c) an augmenting path α in R_0 (indicated by hollow arrows). (d) two new edge-disjoint paths π_1 and π_2 , reconstructed from π_0 and α (both indicated by bold arrows).

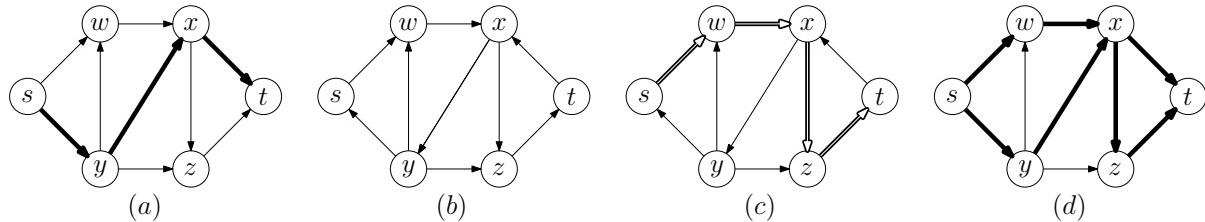


Figure 4: The residual digraph of Figure 3 with another augmenting path and two new paths sets, $\{s.w.x.t, s.y.x.z.t\}$, $\{s.w.x.z.t, s.y.x.t\}$, which are edge-disjoint but not node-disjoint.

The pseudo-code of Algorithm 6 effectively finds the maximum number (and a representative set) of edge-disjoint paths from s and t .

Algorithm 6 (Basic edge-disjoint paths algorithm).

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1 Input: a digraph  $G = (V, E)$  and two nodes  $s \in V, t \in V$ 
2  $k = 0$  (the stage counter)
3  $P_0 = \emptyset$  (the current set of edge-disjoint paths)
4  $R_0 = G$  (the current residual digraph)
5 loop
6    $\alpha =$  an augmenting path in  $R_k$ , from  $s$  to  $t$ , if any (this is a search operation)
7   if  $\alpha = \text{null}$  then break
8    $k = k + 1$  (next stage)
9    $P_k =$  the larger paths set constructed using  $P_{k-1}$  and  $\alpha$  (as indicated in Fact 5)
10   $R_k =$  the residual digraph of  $G$  and  $P_k$ 
11 end loop
12 Output:  $k$  and  $P_k$ , i.e. the maximum number of edge-disjoint paths and a representative set

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Typically, the internal implementation of search at step 6 alternates between a *forward* mode in the residual digraph, which tries to extend a partial augmenting path, and a backwards *backtrack* mode in the residual digraph, which retreats from an unsuccessful attempt, looking for other ways to move forward. The internal implementation of step 9 (i.e. Fact 5) walks backwards in the residual digraph, as a *consolidation* phase, which recombines the newly found augmenting path with the existing edge-disjoint paths.

This algorithm runs in $k + 1$ stages, i.e. in up to $\text{outdegree}(s) + 1$ stages, if we count the number of times it looks for an augmenting paths, and terminates when a new augmenting path is not found. The actual procedure used (in step 6) to find the augmenting path separates two families of algorithms: (1) algorithms from the Ford-Fulkerson family use a depth-first-search (DFS); (2) algorithms from the Edmonds-Karp family use a breadth-first-search (BFS). As usual, both DFS and BFS use “bread crumb” objects, as markers, to avoid cycles; at the end of each stage, these markers are cleaned, to start again with a fresh context. In this paper, we develop P algorithms from the Ford-Fulkerson family, i.e. using DFS.

3.2 Node disjoint paths in digraphs

The edge-disjoint version can be also used to find node-disjoint paths. The textbook solution for the node-disjoint problem is usually achieved by a simple procedure which transforms the original digraph in such a way that, on the transformed digraph, the edge-disjoint problem is identical to the node-disjoint problem of the original digraph. Essentially, this procedure globally replaces every node v , other than s and t , with two nodes, an *entry* node v_1 and an *exit* node v_2 , connected by a single arc (v_1, v_2) . More formally, the new digraph $G' = (V', E')$ has $V' = \{s, t\} \cup \{v_1, v_2 \mid v \in V \setminus \{s, t\}\}$, $E' = \{(v_1, v_2) \mid v \in V \setminus \{s, t\}\} \cup \{(u_2, v_1) \mid (u, v) \in E\}$, where, for convenience, we assume that $s_1 = s_2 = s$ and $t_1 = t_2 = t$ are aliases. This standard node-splitting technique is illustrated in Figure 5. It is straightforward to see that the newly introduced arcs (w_1, w_2) , (x_1, x_2) , (y_1, y_2) and (z_1, z_2) , constrain any edge-disjoint solution to be also node-disjoint.

However, in our case, since each node is identified with a P systems cell, we cannot solve the node-disjoint paths problem using the standard node-splitting technique. We propose two non-standard search

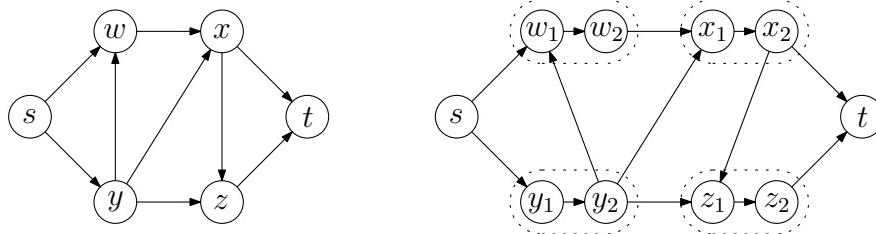
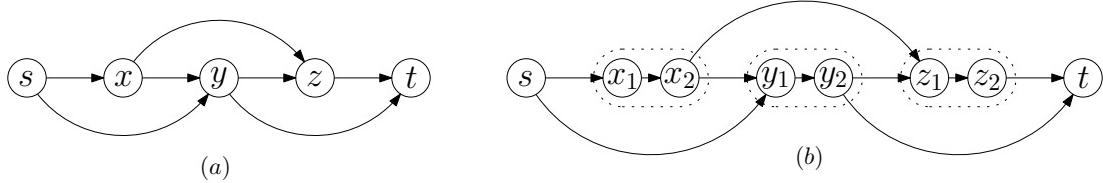


Figure 5: The node splitting technique.

rules, which together limit the out-flow capacity of each $v \in V \setminus \{s, t\}$ to one, by simulating the node-splitting technique, without actually splitting the nodes. We believe that our rules can be used in other distributed network models where the standard node-splitting technique is not applicable. These rules are illustrated by the scenario presented in Figure 6, where we assume that we have already determined a first flow-path, $s.x.y.z.t$, and we are now trying to build a new augmenting path.

1. Consider the case when the augmenting path, consisting of s , tries flow-node y via the non-flow arc (s, y) . We cannot continue with the existing non-flow arc (y, t) (as the edge-disjoint version would do), because this will exceed node y 's capacity, which is one already. Therefore, we continue the search with just the reversed flow-arc (y, x) . Note that, in the underlying node-splitting scenario, we are only visiting the entry node y_1 , but not its exit pair y_2 .
2. Consider now the case when the augmenting path, extended now to $s.y.x.z$, tries again the flow-node y , via the reversed flow-arc (z, y) . It may appear that we are breaking the traditional search rules, by re-visiting the already visited node y . However, there is no infringement in the underlying node-splitting scenario, where we are now trying the not-yet-visited exit node y_2 (to extend the underlying augmenting path $s.y_1.x_2.z_1$). From y , we continue with any available non-flow arc, if any, otherwise, we backtrack. In our example, we continue with arc (y, t) . We obtain a new augmenting “path”, $s.y.x.z.y.t$ (corresponding to the underlying augmenting path $s.y_1.x_2.z_1.y_2.t$). We further recombine it with the already existing flow-path $s.x.y.z.t$, and we finally obtain the two possible node-disjoint paths, $s.x.z.t$ and $s.y.t$.

Figure 6: Node-disjoint paths. (a) non-standard search: flow path $s.x.y.z.t$ and augmenting “path” $s.y.x.z.y.t$. (b) node-splitting: flow path $s.x_1.x_2.y_1.y_2.z_1.z_2.t$ and augmenting path $s.y_1.x_2.z_1.y_2.t$.

The following theorem is now straightforward:

Theorem 7. *If the augmented path search in step 6 of Algorithm 6 is modified as indicated above, the algorithm will terminate with a restricted subset of edge-disjoint paths, forming a maximum cardinal subset of node-disjoint paths.*

3.3 Pointer management

With respect to the implementation, the edge-disjoint version provides its own additional challenge, not present in the node-disjoint version. In the node-disjoint version, each flow-node needs only one pointer to its flow-predecessor and another to its flow-successor. However, in the edge-disjoint version, a flow-node can have k flow-predecessors and k flow-successors, with $k \geq 1$, where each combination is possible, giving rise to $k!$ different edge-disjoint paths sets, each of size k , passing through this node. A naive approach would require recording full details of all $k!$ possible size- k paths sets, or, at least, full details for one of them.

In our simplified approach, we do not keep full path details; instead, a node needs only two size k lists: its flow-predecessors list and its flow-successors list. Using this information, any of the actual $k!$ paths sets can be recreated on the fly, by properly matching flow-predecessors with flow-successors. As an example, consider node x of Figure 4 (d), which has two flow-predecessors, w and y , and two flow-successors, t and z ; thus w is part of four distinct paths. Node w needs only two size- k lists: its flow-predecessors list, $\{w, y\}$, and its flow-successors list $\{z, t\}$.

3.4 Possible optimization

We propose a potential speed-up for Algorithm 6. We restrict this discussion to edge-disjoint paths and standard DFS; however, the discussion can be generalized to more general flows and other search patterns. Consider $V_s = E \cap (\{s\} \times V) = \{(s, v_1), (s, v_2), \dots, (s, v_{d_s})\}$, where $d_s = \text{outdegree}(s)$. Step 6 systematically tries all arcs in V_s . Without loss of generality, we assume that step 6 always tries arcs in the order $(s, v_1), (s, v_2), \dots, (s, v_{d_s})$, stopping upon the first arc which is identified as starting an augmenting path α , say (s, v_r) , where $r \in [1, d_s]$. For brevity, we will indicate this by saying that arc (s, v_r) is the *first* one that *succeeds* (and arcs $(s, v_1), (s, v_2), \dots, (s, v_{r-1})$ *fail*).

Consider a complete run of Algorithm 6. Assume that this algorithm finds k augmenting paths and then stops. Assume that stage $j \in [1, k]$, finds a new augmenting path α_j , which starts with arc (s, v_{i_j}) , i.e. arc (s, v_{i_j}) is the first one that succeeds at stage j . A direct implementation of Algorithm 6 seems to require that step 6 starts a completely new search for each stage, restarting from (s, v_1) and retrying arcs that have been previously tried (whether they failed or succeeded).

However, this is *not* necessary. Theorem 8 indicates that stage $j+1$ does not need to retry the nodes that have already been considered (whether they failed or succeeded). Specifically, the indices indicating the successful arcs in V_s are ordered by stage number, $i_1 < i_2 < \dots < i_k$, and, at stage $j \in [1, k]$, we can start the new search directly from arc $(s, v_{i_{j-1}+1})$ (where $i_0 = 0$).

Theorem 8. *Step 6 of Algorithm 6 can start directly with the arc in V_s which follows the previous stage's succeeding arc in V_s .*

Proof. At stage $j \in [1, k]$, after finding augmenting path α_j , the algorithm fragments α_i together with the previous set of edge-disjoint paths, P_i , deletes some arcs and reassembles a new and larger set of edge-disjoint-paths, P_{i+1} , such that $|P_{i+1}| = |P_i| + 1$.

(1) Consider arc (s, v_{i_j}) , the starting arc of path α_j . We first show that a successful arc, such as (s, v_{i_j}) , need not be tried again by step 6, for two arguments:

(1.1) Arc (s, v_{i_j}) cannot start another augmenting path, because all following residual digraphs will only contain its reversal (v_{i_j}, s) , never its direct form (s, v_{i_j}) .

(1.2) Arc (s, v_{i_j}) cannot be revisited as part of another augmenting path. No augmenting path contains arcs from $V \times \{s\}$, because the search (DFS or BFS) avoids already visited nodes (marked with

“pebbles”), and s is always the starting point and thus the first node marked. Therefore, once successful arc (s, v_{i_j}) is never deleted by “flow pushback operations”.

From (1.1) and (1.2), we conclude that, once successful, an arc in V_s will always be the starting arc of an edge-disjoint path and need not be tried again by step 6.

(2) We next show that, once failed, an arc in V_s will always fail. This part of the proof is by contradiction. We select the first arc that succeeds after first failing and we exhibit a contradiction. Consider that arc (s, v_{i_g}) is this arc, which succeeded at stage g , but failed at least one earlier stage, and let f be the earliest such stage, $f < g$. It is straightforward to see that, in this case, $i_1 < i_2 < \dots < i_{f-1} < i_g < i_f < i_{f+1} < \dots < i_{g-1}$, and arc (s, v_{i_g}) was tried and failed at all stages between f (inclusive) and g .

As a thought experiment, let us stop the algorithm after step g . We have obtained g augmenting paths, thus a set P_g , of g edge-disjoint paths, starting with arcs $(s, v_1), (s, v_2), \dots, (s, v_{f-1}), (s, v_f), (s, v_{f+1}), \dots, (s, v_g)$. Following the same though experiment, let us run the algorithm on digraph G' , obtained from G , by deleting all arcs in V_s except arc (s, v_{i_g}) and those arcs that have been successful, before arc (s, v_{i_g}) was first tried and failed. More formally, $G' = (V, E')$, where $E' = E \setminus (V \setminus V'_s)$, $V'_s = \{(s, v_1), (s, v_2), \dots, (s, v_{f-1}), (s, v_g)\}$. Obviously, $|V'_s| = f$ and digraph G' admits exactly f edge-disjoint paths, because (a) each of the remaining arcs in V'_s can be the start of an edge-disjoint path in P_g (which do not use any other arc in V_s), and (b) digraph G' cannot admit more than $|V'_s| = f$ edge-disjoint paths.

It is straightforward to see that, Algorithm 6, running on digraph G' , will follow exactly the same steps as running on digraph G , up to the point when it first fails on arc (s, v_{i_g}) . At this point, the run on digraph G' stops, after finding f augmenting paths and constructing f edge-disjoint paths.

Thus, the algorithm fails, because $f < g$, which contradicts its correctness. Therefore, the algorithm will never succeed on an arc that has already failed and never needs reconsidering again such arcs.

This completes the second part of the proof. \square

4 Structural and search digraphs in P systems

In this section, we look at various way to reformulate the digraph edge- and node-disjoint path problems as a native P system problem. The P system we consider is “physically” based on a digraph, but this digraph is not necessarily the *virtual* search digraph $G = (V, E)$, on which we intend to find edge- and node-disjoint paths. Given a simple P system $\Pi = (O, K, \delta)$, where δ is its structural digraph, we first identify cells as nodes of interest, $V \simeq K$. However, after that, we see three fundamentally distinct scenarios, which differ in the way how the *forward* and *backward* modes (i.e. *backtrack* and *consolidation*) of Algorithm 6 map to the residual arcs and finally to the structural arcs.

1. We set $E \simeq \delta$. In this case, the forward mode follows the direction of parent-child arcs of δ , while the backward modes follow the reverse direction, from child to parent.
2. We set $E \simeq \{(v, u) \mid (u, v) \in \delta\}$. In this case, the the backward modes follow the direction of parent-child arcs of δ , while the forward mode of the search follows the reverse direction, from child to parent.
3. We set $E \simeq \{(u, v), (v, u) \mid (u, v) \in \delta\}$. In this case, the resulting search digraph is symmetric, and each of the arcs followed by the forward or backward modes of the search can be either a parent-child arc in the original δ or its reverse.

Cases (1) and (2) are simpler to develop. However, in this paper, we look for solutions in case (3), where all messages must be sent to all neighbors, parents and children together. Therefore, our rewriting

rules use the $\beta = \downarrow$ and $\gamma \in \{\text{rep}1\} \cup K$ transfer operators (also indicated in Section 2). Figure 7 illustrates a simple P module and these three scenarios.

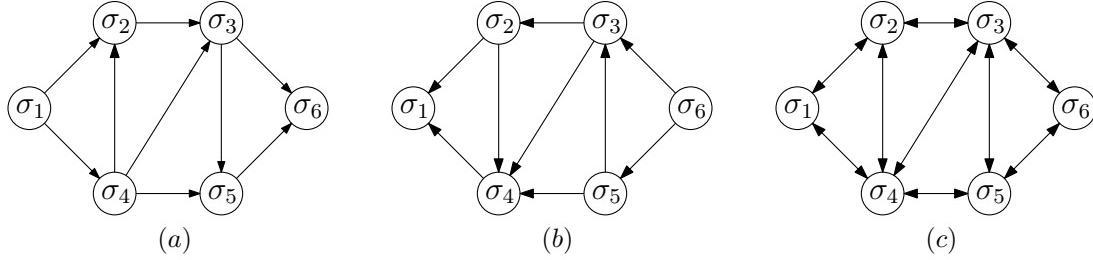


Figure 7: Three virtual search digraphs for the same simple P module. (a) Same “physical” and search structure. (b) The search structure reverses the “physical” structure. (c) The search structure covers both the “physical” structure and its reverse.

Note that, in any of the three cases, Algorithm 6 needs to be able to follow both the parent-child and the child-parent directions of P system structure. Therefore, the structural arcs must support duplex communication channels.

After fixing the directions used by the virtual graph G , the next problem is to let the nodes identify their neighbors, i.e. discover the local network topology.

5 Discovering cell neighbors

In this phase, cells discover their own neighbors. Essentially, each cell sends its own ID to all its neighbors and records the IDs sent from its neighbors. This is a preliminary phase which is identical, for both edge-disjoint and node-disjoint versions. Table 8 illustrates the immediate neighborhoods, discovered at the end of this phase, for the P system of Figure 7 (a), with the virtual search structure shown in Figure 7 (c).

Table 8: Neighbors table for the P system of Figure 7 (a). An object n_j indicates that cell σ_j is a neighbor of the current cell.

Cell	Neighbors	Objects
σ_1	$\{\sigma_2, \sigma_4\}$	$\{n_2, n_4\}$
σ_2	$\{\sigma_1, \sigma_3, \sigma_4\}$	$\{n_1, n_3, n_4\}$
σ_3	$\{\sigma_2, \sigma_4, \sigma_5, \sigma_6\}$	$\{n_2, n_4, n_5, n_6\}$
σ_4	$\{\sigma_1, \sigma_2, \sigma_3, \sigma_5\}$	$\{n_1, n_2, n_3, n_5\}$
σ_5	$\{\sigma_3, \sigma_4, \sigma_6\}$	$\{n_3, n_4, n_6\}$
σ_6	$\{\sigma_3, \sigma_5\}$	$\{n_3, n_5\}$

The set of objects used in this phase is $\{a, k, z\} \cup \bigcup_{1 \leq j \leq n} \{g_j, u_j, n_j\}$. These objects have the following meanings: a indicates a cell reachable from σ_s ; k is the marker of the source cell; z is the marker of the target cell; n_j indicates that σ_j is a neighbor of the current cell; g_j, u_j indicate that σ_j is the target cell; g_j only appears in the source cell, while u_j does not have this restriction.

Initially, the source cell σ_s has one copy of g_j , representing the ID of the target cell σ_j , and the other cells are empty. All cells start in state s_0 . Each reachable cell progresses through states s_0, s_1, s_2, s_3, s_4 , according to the rules given below. In these generic rules (as elsewhere in this paper), we implicitly assume that (1) subscript $i \in \{1, 2, \dots, n\}$ is customized for each cell to its cell ID; and (2) subscript j runs over all cell IDs ($j \in \{1, 2, \dots, n\}$), effectively instantiating n versions of each generic rule where it appears.

0. Rules for state s_0 :

- 1 $s_0 g_i \rightarrow_{\min} s_0$
- 2 $s_0 g_j \rightarrow_{\min} s_1 ak(u_j) \downarrow_{\text{rep1}}$
- 3 $s_0 u_i \rightarrow_{\min} s_1 az(u_i) \downarrow_{\text{rep1}}$
- 4 $s_0 u_i \rightarrow_{\max} s_1$
- 5 $s_0 u_j \rightarrow_{\min} s_1 a(u_j) \downarrow_{\text{rep1}}$

1. Rules for state s_1 :

- 1 $s_1 a \rightarrow_{\min} s_2 a(n_i) \downarrow_{\text{rep1}}$

2. Rules for state s_2 :

- 1 $s_2 a \rightarrow_{\min} s_3 a$

3. Rules for state s_3 :

- 1 $s_3 a \rightarrow_{\min} s_4 a$
- 2 $s_3 u_j \rightarrow_{\max} s_4$

The following example indicates how our generic rules are instantiated to take account the cell IDs, more specifically, how rules 0.1 and 0.2 are instantiated in cell σ_1 :

- $s_0 g_1 \rightarrow_{\min} s_0$
- $s_0 g_1 \rightarrow_{\min} s_1 ak(u_1) \downarrow_{\text{rep1}}$
- $s_0 g_2 \rightarrow_{\min} s_1 ak(u_2) \downarrow_{\text{rep1}}$
- $\dots,$
- $s_0 g_n \rightarrow_{\min} s_1 ak(u_n) \downarrow_{\text{rep1}}$

The state transitions performed by cell σ_i , $i \in \{1, 2, \dots, n\}$, are briefly discussed below.

- $s_0 \rightarrow s_1$: If σ_i contains g_j , σ_i becomes the *source cell*. The source cell broadcasts object u_j to all its neighbors. After receiving an object u_j , cell σ_i becomes either the *target cell*, if $i = j$; or an *intermediate cell*, otherwise. Further, each cell relays one of received u_j objects to all its neighbors.
- $s_1 \rightarrow s_2$: Cell σ_i broadcasts n_i to all its neighbors. Additionally, σ_i accumulates n_j objects from neighbors.
- $s_2 \rightarrow s_3$: Cell σ_i accumulates further n_j objects from neighbors.
- $s_3 \rightarrow s_4$: Cell σ_i accumulates further n_j objects from neighbors. Additionally, σ_i removes superfluous e_j objects.

6 Simple P module rules for edge-disjoint paths algorithm

First, we give a simple P module specification of the edge-disjoint paths algorithm presented in Section 3. We explicitly state our problem in terms of expected input and output. We need to compute a set of edge-disjoint paths of maximum cardinality between given source and target cells.

Edge-Disjoint Paths Problem

Input: A simple P module $\Pi = (O, K, E, \delta)$, where the source cell $\sigma_s \in K$ contains a token t_t identifying the ID of the target cell $\sigma_t \in K$.

Output: If $s \neq t$, each cell $\sigma_i \in K$ contains a set of objects $P_i = \{p_j \mid (j, i) \text{ is a flow-arc}\}$ and a set of

objects $C_i = \{c_j \mid (i, j) \text{ is a flow-arc}\}$ that represent a maximum set of edge-disjoint paths from σ_s to σ_t , where the following constraints hold:

flow-arcs: $c_i \notin C_i, p_i \notin P_i, c_j \in C_i \Leftrightarrow p_i \in P_j$ and $c_j \in C_i \Rightarrow j \in \delta(i) \cup \delta^{-1}(i)$.

source and target: $P_s = \emptyset$ and $C_t = \emptyset$.

in flow = out flow: If $i \notin \{s, t\}$ then $|C_i| = |P_i|$.

only paths: With $S(I) = \bigcup_{i \in I} \{j \mid j \in C_i\}$, $S^{n-1}(I) = S(S(\cdots S(I) \cdots)) = \emptyset$.

Because of the network flow properties, we must also have $|C_s| = |P_t|$, which also represents the maximum number of edge-disjoint paths.

This implementation has two phases: Phase I, which is the discovery phase described in Section 5 (using states s_0 to s_4), and Phase II, described below (which starts in state s_4 and ends in state s_{13}). Table 9 illustrates the expected algorithm output, for a simple P module with the cell structure corresponding to Figure 7 (a). For convenience, although these are deleted near the algorithm's end, we also list all the local neighborhood objects $N_i = \{n_j \mid j \in \delta(i) \cup \delta^{-1}(i)\}$, for $i \in \{1, 2, \dots, n\}$, which are determined in Phase I.

Table 9: A representation of maximum edge-disjoint paths for simple P module of Figure 7 (a).

Cell \ Objects	N_i	P_i	C_i
σ_1	$\{n_2, n_3\}$	\emptyset	$\{c_2, c_4\}$
σ_2	$\{n_1, n_3, n_4\}$	$\{p_1\}$	$\{c_3\}$
σ_3	$\{n_2, n_4, n_5, n_6\}$	$\{p_2, p_4\}$	$\{c_5, c_6\}$
σ_4	$\{n_1, n_2, n_3, n_5\}$	$\{p_1\}$	$\{c_3\}$
σ_5	$\{n_3, n_4, n_6\}$	$\{p_3\}$	$\{c_6\}$
σ_6	$\{n_3, n_5\}$	$\{p_3, p_5\}$	\emptyset

In addition to the set of objects used in Phase I, this phase uses the following set of objects: $\{b_j, c_j, d_j, e_j, f_j, h_j, m_j, p_j, q_j, r_j, t_j, x_j, y_j\} \cup \{v, w\}$. In cell σ_i , these objects have the following meanings:

- b_j indicates a pushback, received from σ_i 's flow-successor σ_j ;
- e_j records the sender of a pushback, if σ_i is not-yet-visited;
- h_j records the sender of a pushback, if σ_i has already been visited;
- c_j indicates that σ_j is σ_i 's flow-successor;
- p_j indicates that σ_j is σ_i 's flow-predecessor;
- r_j records a pushback sent by σ_i to its flow-predecessor σ_j ;
- t_j records a backtrack request, after a failed pushback to σ_j ;
- d_j indicates that σ_j is σ_i 's search-successor;
- q_j indicates that σ_j is σ_i 's search-predecessor;
- f_j indicates an attempted search extension received from σ_j ;
- m_j records that σ_j was unsuccessfully tried;
- x_j indicates that σ_j rejected a flow-extension or flow-pushback attempt;

- x_j indicates that σ_j accepted a flow-extension or flow-pushback attempt;
- v requests that σ_i resets the record of all tried and visited neighbors;
- w requests that σ_i remains idle for one step.

Initially, the source cell σ_s has one copy of g_j , representing the ID of the target cell σ_j , and the other cells are empty. All cells start in state s_0 . According to the rules of Phase I, each reachable cell progresses to state s_4 , which is the start of Phase II, whose generic rules are given below. In these rules (as in Phase I), we implicitly assume that (1) subscript $i \in \{1, 2, \dots, n\}$ is customized for each cell to its cell ID; and (2) subscripts j, k run over all cell IDs, $j, k \in \{1, 2, \dots, n\}$, and $j \neq k$. To apply the *optimization* proposed in Section 3.4, replace rule 5.3 “ $s_5 d_j x_j \rightarrow_{\min} s_5 am_j$ ” by “ $s_5 d_j x_j \rightarrow_{\min} s_5 a$ ”.

4. Rules for a cell σ_i in state s_4 :

- 1 $s_4 k \rightarrow_{\min} s_5 k$
- 2 $s_4 z \rightarrow_{\min} s_6 z$
- 3 $s_4 a \rightarrow_{\min} s_7 a$

5. Rules for a cell σ_i in state s_5 :

- 1 $s_5 an_j \rightarrow_{\min} s_5 d_j (f_i)_{\downarrow j}$
- 2 $s_5 d_j y_j \rightarrow_{\min} s_{12} ac_j ww (v)_{\downarrow_{\text{rep1}}}$
- 3 $s_5 d_j x_j \rightarrow_{\min} s_5 am_j$
- 4 $s_5 b_j \rightarrow_{\min} s_5 (x_i)_{\downarrow j}$
- 5 $s_5 f_j \rightarrow_{\min} s_5 (x_i)_{\downarrow j}$
- 6 $s_5 ak \rightarrow_{\min} s_{13} aaww (a)_{\downarrow_{\text{rep1}}}$

6. Rules for a cell σ_i in state s_6 :

- 1 $s_6 n_j f_j \rightarrow_{\min} s_6 p_j (y_i)_{\downarrow j}$
- 2 $s_6 v \rightarrow_{\min} s_{12} ww (v)_{\downarrow_{\text{rep1}}}$
- 3 $s_6 aaz \rightarrow_{\min} s_{13} aaww (a)_{\downarrow_{\text{rep1}}}$

7. Rules for a cell σ_i in state s_7 :

- 1 $s_7 v \rightarrow_{\min} s_{12} ww (v)_{\downarrow_{\text{rep1}}}$
- 2 $s_7 aa \rightarrow_{\min} s_{13} aaww (a)_{\downarrow_{\text{rep1}}}$
- 3 $s_7 n_j f_j \rightarrow_{\min} s_8 q_j$
- 4 $s_7 c_j b_j \rightarrow_{\min} s_8 e_j$
- 5 $s_7 h_j \rightarrow_{\min} s_{11} c_j (x_i)_{\downarrow j}$
- 6 $s_7 p_j q_k \rightarrow_{\min} s_{10} p_j q_k$
- 7 $s_7 q_j \rightarrow_{\min} s_7 m_j (x_i)_{\downarrow j}$
- 8 $s_7 f_j \rightarrow_{\min} s_7 (x_i)_{\downarrow j}$

8. Rules for a cell σ_i in state s_8 :

- 1 $s_8 an_j \rightarrow_{\min} s_9 ad_j (f_i)_{\downarrow j}$
- 2 $s_8 a \rightarrow_{\min} s_{10} a$

9. Rules for a cell σ_i in state s_9 :

- 1 $s_9 d_j y_j e_k \rightarrow_{\min} s_7 c_j m_k (y_i)_{\downarrow k}$
- 2 $s_9 d_j y_j q_k \rightarrow_{\min} s_7 c_j p_k (y_i)_{\downarrow k}$
- 3 $s_9 d_j x_j \rightarrow_{\min} s_8 m_j$
- 4 $s_9 c_j b_j \rightarrow_{\min} s_9 c_j (x_i)_{\downarrow j}$
- 5 $s_9 n_j f_j \rightarrow_{\min} s_9 m_j (x_i)_{\downarrow j}$

10. Rules for a cell σ_i in state s_{10} :

- 1 $s_{10} ap_j \rightarrow_{\min} s_{11} ar_j (b_i)_{\downarrow j}$
- 2 $s_{10} ae_j \rightarrow_{\min} s_7 ac_j (x_i)_{\downarrow j}$
- 3 $s_{10} aq_j \rightarrow_{\min} s_7 am_j (x_i)_{\downarrow j}$

11. Rules for a cell σ_i in state s_{11} :

- 1 $s_{11} r_j y_j e_k \rightarrow_{\min} s_7 m_j m_k (y_i)_{\downarrow k}$
- 2 $s_{11} r_j y_j q_k \rightarrow_{\min} s_7 m_j p_k (y_i)_{\downarrow k}$
- 3 $s_{11} r_j x_j \rightarrow_{\min} s_{10} t_j$
- 4 $s_{11} c_j b_j \rightarrow_{\min} s_7 h_j$
- 5 $s_{11} n_j f_j \rightarrow_{\min} s_{11} m_j (x_i)_{\downarrow j}$

12. Rules for a cell σ_i in state s_{12} :

- 1 $s_{12} w \rightarrow_{\min} s_{12}$
- 2 $s_{12} v \rightarrow_{\max} s_{12}$
- 3 $s_{12} m_j \rightarrow_{\min} s_{12} n_j$
- 4 $s_{12} t_j \rightarrow_{\min} s_{12} p_j$
- 5 $s_{12} k \rightarrow_{\min} s_5 k$
- 6 $s_{12} z \rightarrow_{\min} s_6 z$
- 7 $s_{12} a \rightarrow_{\min} s_7 a$

13. Rules for a cell σ_i in state s_{13} :

- 1 $s_{13} w \rightarrow_{\min} s_{13}$
- 2 $s_{13} a \rightarrow_{\max} s_0$
- 3 $s_{13} t_j \rightarrow_{\min} s_0 p_j$
- 4 $s_{13} n_j \rightarrow_{\min} s_0$
- 5 $s_{13} m_j \rightarrow_{\min} s_0$

The following paragraphs describe details of several critical steps of our edge-disjoint algorithm, such as forward and consolidation modes of intermediate cells.

- The rules in state s_8 cover the forward mode attempt of an intermediate cell σ_i , via a non-flow arc. If σ_i has a not-yet-tried neighbor $\sigma_h \notin P_i \cup C_i$, the search continues with σ_h ; otherwise the search backtracks.
- The rules in state s_9 cover the consolidation mode for an intermediate cell σ_i , who has succeeded a forward extension on a non-flow arc.

During the consolidation process, the behavior of σ_i depends on whether σ_i was reached by a flow arc or non-flow arc:

- If σ_i was reached by flow arc (i, j) , in a reverse direction, σ_i replaces its flow-successor σ_j with σ_h , where σ_h is its search-predecessor.
- If σ_i was reached by a non-flow arc (k, i) , σ_i sets σ_k as a flow-predecessor and σ_h as a flow-successor, where σ_h and σ_j are σ_i 's search-predecessor and search-successor, respectively.
- The rules in state s_{10} cover the forward mode attempt of an intermediate cell σ_i , via a pushback. If σ_i has a not-yet-tried flow-predecessor σ_h , the search continues with σ_h (i.e. use flow arc (h, i) , in a reverse direction); otherwise the search backtracks.
- The rules in state s_{11} cover the consolidation mode for an intermediate cell σ_i , who has succeeded a forward extension via a pushback to σ_h (i.e. on a flow-arc (h, i) , in a reverse direction).

During the consolidation process, the behavior of σ_i depends on whether σ_i was reached by a flow arc or non-flow arc:

- If σ_i was reached by flow arc (i, j) , in a reverse direction, σ_i removes its flow-predecessor σ_h and flow-successor σ_j .
- If σ_i was reached by a non-flow arc (k, i) , σ_i replaces its flow-predecessor σ_h with σ_k , where σ_k is σ_i 's search-predecessor.

Theorem 9. *For a simple P module with n cells and $m = |\delta|$ edges, the algorithm in this section runs in $O(mn)$ steps.*

7 Simple P module rules for node-disjoint paths algorithm

First, we give a simple P module specification of the node-disjoint paths algorithm presented in Section 3. We explicitly state our problem in terms of expected input and output. We need to compute a set of node-disjoint paths of maximum cardinality between given source and target cells.

Node-Disjoint Paths Problem

Input: A simple P module $\Pi = (O, K, E, \delta)$, where the source cell $\sigma_s \in K$ contains a token t_t identifying the ID of the target cell $\sigma_t \in K$.

Output: If $s \neq t$, each cell $\sigma_i \in K$ contains a set of objects $P_i = \{p_j \mid (j, i) \text{ is a flow-arc}\}$ and a set of objects $C_i = \{c_j \mid (i, j) \text{ is a flow-arc}\}$ that represent a maximum set of node-disjoint paths from σ_s to σ_t where the following constraints hold:

flow-arcs: $c_i \notin C_i, p_i \notin P_i, c_j \in C_i \Leftrightarrow p_i \in P_j$ and $c_j \in C_i \Rightarrow j \in \delta(i) \cup \delta^{-1}(i)$.

source and target: $P_s = \emptyset$ and $C_t = \emptyset$.

node-disjoint: If $i \notin \{s, t\}$ then $|C_i| = |P_i| \leq 1$.

only paths: With $S(i) = \begin{cases} t & \text{if } i \in \{s, t\} \text{ or } |C_i| = 0 \\ j & \text{when } C_i = \{c_j\} \end{cases}$, $S^{n-1}(i) = S(S(\dots S(i) \dots)) = t$.

Because of the network flow properties, we must also have $|C_s| = |P_t|$, which also represents the maximum number of node-disjoint paths. Notice the constraints to require only paths has been simplified in that the successor $S(i)$ of non-source cell σ_i is a single cell instead of a set of cells that was needed for the general edge-disjoint problem.

Table 10 illustrates the expected algorithm output, for a simple P module with the cell structure corresponding to Figure 7 (a). For convenience, although these are deleted near the algorithm's end, we also list all the local neighborhood objects $N_i = \{n_j \mid j \in \delta(i) \cup \delta^{-1}(i)\}$, for $i \in \{1, 2, \dots, n\}$, which are determined in Phase I.

Table 10: A representation of maximum node-disjoint paths for simple P module of Figure 7 (a).

Cell \ Objects	N_i	P_i	C_i
σ_1	$\{n_2, n_3\}$	\emptyset	$\{c_2, c_4\}$
σ_2	$\{n_1, n_3, n_4\}$	$\{p_1\}$	$\{c_3\}$
σ_3	$\{n_2, n_4, n_5, n_6\}$	$\{p_2\}$	$\{c_6\}$
σ_4	$\{n_1, n_2, n_3, n_5\}$	$\{p_1\}$	$\{c_5\}$
σ_5	$\{n_3, n_4, n_6\}$	$\{p_4\}$	$\{c_6\}$
σ_6	$\{n_3, n_5\}$	$\{p_3, p_5\}$	\emptyset

The rules of this node-disjoint path algorithm are exactly the rules of the edge-disjoint path algorithm described in Section 6, where the rules for state s_7 are replaced by the following group of rules.

7. Rules for a cell σ_i in state s_7 :

- 1 $s_7 v \rightarrow_{\min} s_{12} ww(v) \downarrow_{\text{rep1}}$
- 2 $s_7 aa \rightarrow_{\min} s_{13} aaww(a) \downarrow_{\text{rep1}}$
- 3 $s_7 n_j f_j p_k \rightarrow_{\min} s_{10} q_j p_k$
- 4 $s_7 n_j f_j \rightarrow_{\min} s_8 q_j$
- 5 $s_7 c_j b_j \rightarrow_{\min} s_8 e_j$
- 6 $s_7 h_j \rightarrow_{\min} s_8 e_j$
- 7 $s_7 r_j \rightarrow_{\min} s_{11} r_j$
- 8 $s_7 q_j \rightarrow_{\min} s_7 m_j(x_i) \uparrow_j$
- 9 $s_7 f_j \rightarrow_{\min} s_7 (x_i) \uparrow_j$

The new state s_7 rules implement our proposed non-standard technique described in Section 3.2, for enforcing node capacities to one, without node-splitting.

The running time of our node-disjoint paths algorithm runs in polynomial number of steps, since the algorithm is direct implementation of Ford-Fulkerson's network flow algorithm.

Theorem 10. *For a simple P module with n cells and $m = |\delta|$ edges, the algorithm in this section runs in $O(mn)$ steps.*

8 Conclusion and Open Problems

Using the newly introduced simple P modules framework, we have presented native P system versions of the edge- and node-disjoint paths problems. We have started from standard network flow ideas, with additional constraints required by our model, e.g., cells that start without any knowledge about the local and global structure. Our P algorithms use a depth-first search technique and iteratively build routing tables, until they find the maximum number of disjoint paths; Our P algorithms run in polynomial time, comparable to the standard versions of the Ford-Fulkerson algorithms. We have proved and used a speedup optimization, which probably was not previously known. For node-disjoint paths, we proposed an alternate set of search rules, which can be used for other synchronous network models, where, as in P systems, the standard node-splitting technique is not applicable.

All of our previous P algorithms assumed that the structural relation δ (of a simple P module) supports duplex communication channels between adjacent P system cells. Substantial modifications are needed when we consider the simplex communication case. It is not just a simple matter of changing the rules of the systems to only follow out-neighbors when we are finding paths from the source to the target—we have explicitly utilized the ability to “push back” flow on a flow-arc, by sending objects to their flow-predecessors, when hunting for augmenting paths. Thus, some new ideas are needed before we can compute disjoint paths when the structural arcs allow only simplex communication.

We also want to know if we can solve the problem of finding disjoint paths between k pairs of $(s_1, t_1) \dots (s_k, t_k)$, that is comparable in performance to the $O(n^3)$ algorithm of [16].

We are interested to know whether a breadth-first search (BFS) approach would be more beneficial than a depth-first search. By using BFS we could potentially exploit more of the parallel nature of P systems.

By combining this paper’s results with our previous P solutions for the Byzantine problem [3, 2], we have now solved one of our original goals, i.e. to solve, where possible, the Byzantine problem for P systems based on general digraphs, not necessarily complete. This more general problem can be solved in two phases: (1) determine all node-disjoint paths in a digraph, assuming that, in this phase, there are no faults; and then (2) solve the consensus problem, even if, in this phase, some nodes fail in arbitrary, Byzantine ways. An interesting problem arises, which, apparently, hasn’t been considered yet. What can we do if the Byzantine nodes already behave in a Byzantine manner in phase (1), while we attempt to build the node-disjoint paths? Can we still determine all or a sufficient number of node-disjoint paths, in the presence of Byzantine faults?

Some of previous experiences [1, 10] have suggested that P systems need to be extended with support for mobile arcs, to incrementally build direct communication channels between originally distant cells and we have offered a preliminary solution. The current experience suggests that this support should be extended to enable straightforward creation of virtual search digraphs on top of existing physical digraphs.

We consider that this continued experience will provide good feedback on the usability of P systems as a formal model for parallel and distributed computing and suggest a range of extensions and improvements, both at the conceptual level and for practical implementations.

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Table 11: Edge-disjoint paths solution traces (steps 0, 1, ..., 30) of the simple P module shown in Figure 7 (a), where σ_1 is the source cell and σ_6 is the target cell.

Step\Cell	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
0	$s_0 g_6$	s_0	s_0	s_0	s_0	s_0
1	$s_1 ak$	$s_0 u_6$	s_0	$s_0 u_6$	s_0	s_0
2	$s_2 a k u_6^2$	$s_1 a n_1 u_6$	$s_0 u_6^2$	$s_1 a n_1 u_6$	$s_0 u_6$	s_0
3	$s_3 a k n_2 n_4 u_6^2$	$s_2 a n_1 n_4 u_6^2$	$s_1 a n_2 n_4 u_6^2$	$s_2 a n_1 n_2 u_6^3$	$s_1 a n_4 u_6$	$s_0 u_6^2$
4	$s_4 a k n_2 n_4$	$s_3 a n_1 n_3 n_4 u_6^2$	$s_2 a n_2 n_4 n_5 u_6^3$	$s_3 a n_1 n_2 n_3 n_5 u_6^3$	$s_2 a n_3 n_4 u_6^2$	$s_1 a n_3 n_5 z$
5	$s_5 a k n_2 n_4$	$s_4 a n_1 n_3 n_4$	$s_3 a n_2 n_4 n_5 n_6 u_6^3$	$s_4 a n_1 n_2 n_3 n_5$	$s_3 a n_3 n_4 n_6 u_6^2$	$s_2 a n_3 n_5 z$
6	$s_5 d_4 k n_2$	$s_7 a n_1 n_3 n_4$	$s_4 a n_2 n_4 n_5 n_6$	$s_7 a f_1 n_1 n_2 n_3 n_5$	$s_4 a n_3 n_4 n_6$	$s_3 a n_3 n_5 z$
7	$s_5 d_4 k n_2$	$s_7 a n_1 n_3 n_4$	$s_7 a n_2 n_4 n_5 n_6$	$s_8 a n_2 n_3 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_4 a n_3 n_5 z$
8	$s_5 d_4 k n_2$	$s_7 a n_1 n_3 n_4$	$s_7 a f_4 n_2 n_4 n_5 n_6$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
9	$s_5 d_4 k n_2$	$s_7 a n_1 n_3 n_4$	$s_8 a n_2 n_5 n_6 q_4$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
10	$s_5 d_4 k n_2$	$s_7 a f_3 n_1 n_3 n_4$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
11	$s_5 d_4 k n_2$	$s_8 a n_1 n_4 q_3$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
12	$s_5 d_4 f_2 k n_2$	$s_9 a d_1 n_4 q_3$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
13	$s_5 d_4 k n_2$	$s_9 a d_1 n_4 q_3 x_1$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
14	$s_5 d_4 k n_2$	$s_8 a m_1 n_4 q_3$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
15	$s_5 d_4 k n_2$	$s_9 a d_4 m_1 q_3$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 f_2 n_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
16	$s_5 d_4 k n_2$	$s_9 a d_4 m_1 q_3 x_4$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
17	$s_5 d_4 k n_2$	$s_8 a m_1 m_4 q_3$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
18	$s_5 d_4 k n_2$	$s_{10} a m_1 m_4 q_3$	$s_9 a d_2 n_5 n_6 q_4$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
19	$s_5 d_4 k n_2$	$s_7 a m_1 m_3 m_4$	$s_9 a d_2 n_5 n_6 q_4 x_2$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
20	$s_5 d_4 k n_2$	$s_7 a m_1 m_3 m_4$	$s_8 a m_2 n_5 n_6 q_4$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_3 n_5 z$
21	$s_5 d_4 k n_2$	$s_7 a m_1 m_3 m_4$	$s_9 a d_6 m_2 n_5 q_4$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a f_3 n_3 n_5 z$
22	$s_5 d_4 k n_2$	$s_7 a m_1 m_3 m_4$	$s_9 a d_6 m_2 n_5 q_4 y_6$	$s_9 a d_3 m_2 n_5 q_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
23	$s_5 d_4 k n_2$	$s_7 a m_1 m_3 m_4$	$s_7 a c_6 m_2 n_5 p_4$	$s_9 a d_3 m_2 n_5 q_1 y_3$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
24	$s_5 d_4 k n_2 y_4$	$s_7 a m_1 m_3 m_4$	$s_7 a c_6 m_2 n_5 p_4$	$s_7 a c_3 m_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
25	$s_{12} a c_4 k n_2 w^2$	$s_7 a m_1 m_3 m_4 v$	$s_7 a c_6 m_2 n_5 p_4$	$s_7 a c_3 m_2 n_5 p_1 v$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
26	$s_{12} a c_4 k n_2 v^2 w$	$s_{12} a m_1 m_3 m_4 v w^2$	$s_7 a c_6 m_2 n_5 p_4 v^2$	$s_{12} a c_3 m_2 n_5 p_1 v w^2$	$s_7 a n_3 n_4 n_6 v$	$s_6 a n_5 p_3 z$
27	$s_{12} a c_4 k n_2$	$s_{12} a n_1 n_3 n_4 v w$	$s_{12} a c_6 m_2 n_5 p_4 v^2 w^2$	$s_{12} a c_3 m_2 n_5 p_1 v^2 w$	$s_{12} a n_3 n_4 n_6 v w^2$	$s_6 a n_5 p_3 v^2 z$
28	$s_5 a c_4 k n_2$	$s_{12} a n_1 n_3 n_4$	$s_{12} a c_6 m_2 n_5 p_4 v w$	$s_{12} a c_3 m_2 n_5 p_1$	$s_{12} a n_3 n_4 n_6 v w$	$s_{12} a n_5 p_3 v w^2 z$
29	$s_5 c_4 d_2 k$	$s_7 a f_1 n_1 n_3 n_4$	$s_{12} a c_6 m_2 n_5 p_4$	$s_7 a c_3 m_2 n_5 p_1$	$s_{12} a n_3 n_4 n_6$	$s_{12} a n_5 p_3 w z$
30	$s_5 c_4 d_2 k$	$s_8 a n_3 n_4 q_1$	$s_7 a c_6 m_2 n_5 p_4$	$s_7 a c_3 m_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_{12} a n_5 p_3 z$

Table 12: Edge-disjoint paths solution traces (steps 31, 32, ..., 61) of the simple P module shown in Figure 7 (a), where σ_1 is the source cell and σ_6 is the target cell.

Step\Cell	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
31	$s_5 c_4 d_2 k$	$s_9 ad_3 n_4 q_1$	$s_7 ac_6 f_2 n_2 n_5 p_4$	$s_7 ac_3 n_2 n_5 p_1$	$s_7 an_3 n_4 n_6$	$s_6 an_5 p_3 z$
32	$s_5 c_4 d_2 k$	$s_9 ad_3 n_4 q_1$	$s_8 ac_6 n_5 p_4 q_2$	$s_7 ac_3 n_2 n_5 p_1$	$s_7 an_3 n_4 n_6$	$s_6 an_5 p_3 z$
33	$s_5 c_4 d_2 k$	$s_9 ad_3 n_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 n_2 n_5 p_1$	$s_7 af_3 n_3 n_4 n_6$	$s_6 an_5 p_3 z$
34	$s_5 c_4 d_2 k$	$s_9 ad_3 n_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 n_2 n_5 p_1$	$s_8 an_4 n_6 q_3$	$s_6 an_5 p_3 z$
35	$s_5 c_4 d_2 k$	$s_9 ad_3 n_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 f_5 n_2 n_5 p_1$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
36	$s_5 c_4 d_2 k$	$s_9 ad_3 n_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_8 ac_3 n_2 p_1 q_5$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
37	$s_5 c_4 d_2 k$	$s_9 ad_3 f_4 n_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_9 ac_3 d_2 p_1 q_5$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
38	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_9 ac_3 d_2 p_1 q_5 x_2$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
39	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_8 ac_3 m_2 p_1 q_5$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
40	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_{10} ac_3 m_2 p_1 q_5$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
41	$s_5 b_4 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_{11} ac_3 m_2 q_5 r_1$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
42	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_{11} ac_3 m_2 q_5 r_1 x_1$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
43	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_{10} ac_3 m_2 q_5 t_1$	$s_9 ad_4 n_6 q_3$	$s_6 an_5 p_3 z$
44	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 m_2 m_5 t_1$	$s_9 ad_4 n_6 q_3 x_4$	$s_6 an_5 p_3 z$
45	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 m_2 m_5 t_1$	$s_8 am_4 n_6 q_3$	$s_6 an_5 p_3 z$
46	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 m_2 m_5 t_1$	$s_9 ad_6 m_4 q_3$	$s_6 af_5 n_5 p_3 z$
47	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2$	$s_7 ac_3 m_2 m_5 t_1$	$s_9 ad_6 m_4 q_3 y_6$	$s_6 ap_3 p_5 z$
48	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1$	$s_9 ac_6 d_5 p_4 q_2 y_5$	$s_7 ac_3 m_2 m_5 t_1$	$s_7 ac_6 m_4 p_3$	$s_6 ap_3 p_5 z$
49	$s_5 c_4 d_2 k$	$s_9 ad_3 m_4 q_1 y_3$	$s_7 ac_5 c_6 p_2 p_4$	$s_7 ac_3 m_2 m_5 t_1$	$s_7 ac_6 m_4 p_3$	$s_6 ap_3 p_5 z$
50	$s_5 c_4 d_2 k y_2$	$s_7 ac_3 m_4 p_1$	$s_7 ac_5 c_6 p_2 p_4$	$s_7 ac_3 m_2 m_5 t_1$	$s_7 ac_6 m_4 p_3$	$s_6 ap_3 p_5 z$
51	$s_{12} ac_2 c_4 k w^2$	$s_7 ac_3 m_4 p_1 v$	$s_7 ac_5 c_6 p_2 p_4$	$s_7 ac_3 m_2 m_5 t_1 v$	$s_7 ac_6 m_4 p_3$	$s_6 ap_3 p_5 z$
52	$s_{12} ac_2 c_4 k v^2 w$	$s_{12} ac_3 m_4 p_1 v w^2$	$s_7 ac_5 c_6 p_2 p_4 v^2$	$s_{12} ac_3 m_2 m_5 t_1 v w^2$	$s_7 ac_6 m_4 p_3 v$	$s_6 ap_3 p_5 z$
53	$s_{12} ac_2 c_4 k$	$s_{12} ac_3 n_4 p_1 v w$	$s_{12} ac_5 c_6 p_2 p_4 v^2 w^2$	$s_{12} ac_3 n_2 n_5 p_1 v^2 w$	$s_{12} ac_6 m_4 p_3 v w^2$	$s_6 ap_3 p_5 v^2 z$
54	$s_5 ac_2 c_4 k$	$s_{12} ac_3 n_4 p_1$	$s_{12} ac_5 c_6 p_2 p_4 v w$	$s_{12} ac_3 n_2 n_5 p_1$	$s_{12} ac_6 n_4 p_3 v w$	$s_{12} ap_3 p_5 v w^2 z$
55	$s_{13} a^2 c_2 c_4 w^2$	$s_7 a^2 c_3 n_4 p_1$	$s_{12} ac_5 c_6 p_2 p_4$	$s_7 a^2 c_3 n_2 n_5 p_1$	$s_{12} ac_6 n_4 p_3$	$s_{12} ap_3 p_5 w z$
56	$s_{13} a^4 c_2 c_4 w$	$s_{13} a^3 c_3 n_4 p_1 w^2$	$s_7 a^3 c_5 c_6 p_2 p_4$	$s_{13} a^3 c_3 n_2 n_5 p_1 w^2$	$s_7 a^2 c_6 n_4 p_3$	$s_{12} ap_3 p_5 z$
57	$s_{13} a^4 c_2 c_4$	$s_{13} a^4 c_3 n_4 p_1 w$	$s_{13} a^4 c_5 c_6 p_2 p_4 w^2$	$s_{13} a^5 c_3 n_2 n_5 p_1 w$	$s_{13} a^3 c_6 n_4 p_3 w^2$	$s_6 a^3 p_3 p_5 z$
58	$s_0 c_2 c_4$	$s_{13} a^4 c_3 n_4 p_1$	$s_{13} a^5 c_5 c_6 p_2 p_4 w$	$s_{13} a^5 c_3 n_2 n_5 p_1$	$s_{13} a^4 c_6 n_4 p_3 w$	$s_{13} a^3 p_3 p_5 w^2$
59	$s_0 c_2 c_4$	$s_0 c_3 p_1$	$s_{13} a^5 c_5 c_6 p_2 p_4$	$s_0 c_3 p_1$	$s_{13} a^4 c_6 n_4 p_3$	$s_{13} a^3 p_3 p_5 w$
60	$s_0 c_2 c_4$	$s_0 c_3 p_1$	$s_0 c_5 c_6 p_2 p_4$	$s_0 c_3 p_1$	$s_0 c_6 p_3$	$s_{13} a^3 p_3 p_5$
61	$s_0 c_2 c_4$	$s_0 c_3 p_1$	$s_0 c_5 c_6 p_2 p_4$	$s_0 c_3 p_1$	$s_0 c_6 p_3$	$s_0 p_3 p_5$

Table 13: Node-disjoint paths solution traces (steps 0, 1, ..., 29) of the simple P module shown in Figure 7 (a), where σ_1 is the source cell and σ_6 is the target cell.

Step\Cell	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
0	$s_0 g_6$	s_0	s_0	s_0	s_0	s_0
1	$s_1 ak$	$s_0 u_6$	s_0	$s_0 u_6$	s_0	s_0
2	$s_2 aku_6^2$	$s_1 an_1u_6$	$s_0 u_6^2$	$s_1 an_1u_6$	$s_0 u_6$	s_0
3	$s_3 akn_2n_4u_6^2$	$s_2 an_1n_4u_6^2$	$s_1 an_2n_4u_6^2$	$s_2 an_1n_2u_6^3$	$s_1 an_4u_6$	$s_0 u_6^2$
4	$s_4 akn_2n_4$	$s_3 an_1n_3n_4u_6^2$	$s_2 an_2n_4n_5u_6^3$	$s_3 an_1n_2n_3n_5u_6^3$	$s_2 an_3n_4u_6^2$	$s_1 an_3n_5z$
5	$s_5 akn_2n_4$	$s_4 an_1n_3n_4$	$s_3 an_2n_4n_5n_6u_6^3$	$s_4 an_1n_2n_3n_5$	$s_3 an_3n_4n_6u_6^2$	$s_2 an_3n_5z$
6	$s_5 d_4kn_2$	$s_7 an_1n_3n_4$	$s_4 an_2n_4n_5n_6$	$s_7 af_1n_1n_2n_3n_5$	$s_4 an_3n_4n_6$	$s_3 an_3n_5z$
7	$s_5 d_4kn_2$	$s_7 an_1n_3n_4$	$s_7 an_2n_4n_5n_6$	$s_8 an_2n_3n_5q_1$	$s_7 an_3n_4n_6$	$s_4 an_3n_5z$
8	$s_5 d_4kn_2$	$s_7 an_1n_3n_4$	$s_7 af_4n_2n_4n_5n_6$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
9	$s_5 d_4kn_2$	$s_7 an_1n_3n_4$	$s_8 an_2n_5n_6q_4$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
10	$s_5 d_4kn_2$	$s_7 af_3n_1n_3n_4$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
11	$s_5 d_4kn_2$	$s_8 an_1n_4q_3$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
12	$s_5 d_4f_2kn_2$	$s_9 ad_1n_4q_3$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
13	$s_5 d_4kn_2$	$s_9 ad_1n_4q_3x_1$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
14	$s_5 d_4kn_2$	$s_8 am_1n_4q_3$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
15	$s_5 d_4kn_2$	$s_9 ad_4m_1q_3$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3f_2n_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
16	$s_5 d_4kn_2$	$s_9 ad_4m_1q_3x_4$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
17	$s_5 d_4kn_2$	$s_8 am_1m_4q_3$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
18	$s_5 d_4kn_2$	$s_{10} am_1m_4q_3$	$s_9 ad_2n_5n_6q_4$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
19	$s_5 d_4kn_2$	$s_7 am_1m_3m_4$	$s_9 ad_2n_5n_6q_4x_2$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
20	$s_5 d_4kn_2$	$s_7 am_1m_3m_4$	$s_8 am_2n_5n_6q_4$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_3n_5z$
21	$s_5 d_4kn_2$	$s_7 am_1m_3m_4$	$s_9 ad_6m_2n_5q_4$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 af_3n_3n_5z$
22	$s_5 d_4kn_2$	$s_7 am_1m_3m_4$	$s_9 ad_6m_2n_5q_4y_6$	$s_9 ad_3m_2n_5q_1$	$s_7 an_3n_4n_6$	$s_6 an_5p_3z$
23	$s_5 d_4kn_2$	$s_7 am_1m_3m_4$	$s_7 ac_6m_2n_5p_4$	$s_9 ad_3m_2n_5q_1y_3$	$s_7 an_3n_4n_6$	$s_6 an_5p_3z$
24	$s_5 d_4kn_2y_4$	$s_7 am_1m_3m_4$	$s_7 ac_6m_2n_5p_4$	$s_7 ac_3m_2n_5p_1$	$s_7 an_3n_4n_6$	$s_6 an_5p_3z$
25	$s_{12} ac_4kn_2w^2$	$s_7 am_1m_3m_4v$	$s_7 ac_6m_2n_5p_4$	$s_7 ac_3m_2n_5p_1v$	$s_7 an_3n_4n_6$	$s_6 an_5p_3z$
26	$s_{12} ac_4kn_2v^2w$	$s_{12} am_1m_3m_4vw^2$	$s_7 ac_6m_2n_5p_4v^2$	$s_{12} ac_3m_2n_5p_1vw^2$	$s_7 an_3n_4n_6v$	$s_6 an_5p_3z$
27	$s_{12} ac_4kn_2$	$s_{12} an_1n_3n_4vw$	$s_{12} ac_6m_2n_5p_4v^2w^2$	$s_{12} ac_3n_2n_5p_1v^2w$	$s_{12} an_3n_4n_6vw^2$	$s_6 an_5p_3v^2z$
28	$s_5 ac_4kn_2$	$s_{12} an_1n_3n_4$	$s_{12} ac_6n_2n_5p_4vw$	$s_{12} ac_3n_2n_5p_1$	$s_{12} an_3n_4n_6vw$	$s_{12} an_5p_3vw^2z$
29	$s_5 c_4d_2k$	$s_7 af_1n_1n_3n_4$	$s_{12} ac_6n_2n_5p_4$	$s_7 ac_3n_2n_5p_1$	$s_{12} an_3n_4n_6$	$s_{12} an_5p_3wz$

Table 14: node-disjoint paths solution traces (steps 30, 31, ..., 59) of the simple P module shown in Figure 7 (a), where σ_1 is the source cell and σ_6 is the target cell.

Step\Cell	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
30	$s_5 c_4 d_2 k$	$s_8 a n_3 n_4 q_1$	$s_7 a c_6 n_2 n_5 p_4$	$s_7 a c_3 n_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_{12} a n_5 p_3 z$
31	$s_5 c_4 d_2 k$	$s_9 a d_3 n_4 q_1$	$s_7 a c_6 f_2 n_2 n_5 p_4$	$s_7 a c_3 n_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
32	$s_5 c_4 d_2 k$	$s_9 a d_3 n_4 q_1$	$s_{10} a c_6 n_5 p_4 q_2$	$s_7 a c_3 n_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
33	$s_5 c_4 d_2 k$	$s_9 a d_3 n_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_7 a b_3 c_3 n_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
34	$s_5 c_4 d_2 k$	$s_9 a d_3 n_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_8 a e_3 n_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
35	$s_5 c_4 d_2 k$	$s_9 a d_3 f_4 n_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_9 a d_2 e_3 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
36	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_9 a d_2 e_3 n_5 p_1 x_2$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
37	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_8 a e_3 m_2 n_5 p_1$	$s_7 a n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
38	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_7 a f_4 n_3 n_4 n_6$	$s_6 a n_5 p_3 z$
39	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 n_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_8 a n_3 n_6 q_4$	$s_6 a n_5 p_3 z$
40	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 f_5 n_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_9 a d_3 n_6 q_4$	$s_6 a n_5 p_3 z$
41	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 m_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_9 a d_3 n_6 q_4 x_3$	$s_6 a n_5 p_3 z$
42	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 m_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_8 a m_3 n_6 q_4$	$s_6 a n_5 p_3 z$
43	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 m_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_9 a d_6 m_3 q_4$	$s_6 a f_5 n_5 p_3 z$
44	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 m_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1$	$s_9 a d_6 m_3 q_4 y_6$	$s_6 a p_3 p_5 z$
45	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 m_5 q_2 r_4$	$s_9 a d_5 e_3 m_2 p_1 y_5$	$s_7 a c_6 m_3 p_4$	$s_6 a p_3 p_5 z$
46	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1$	$s_{11} a c_6 m_5 q_2 r_4 y_4$	$s_7 a c_5 m_2 m_3 p_1$	$s_7 a c_6 m_3 p_4$	$s_6 a p_3 p_5 z$
47	$s_5 c_4 d_2 k$	$s_9 a d_3 m_4 q_1 y_3$	$s_7 a c_6 m_4 m_5 p_2$	$s_7 a c_5 m_2 m_3 p_1$	$s_7 a c_6 m_3 p_4$	$s_6 a p_3 p_5 z$
48	$s_5 c_4 d_2 k y_2$	$s_7 a c_3 m_4 p_1$	$s_7 a c_6 m_4 m_5 p_2$	$s_7 a c_5 m_2 m_3 p_1$	$s_7 a c_6 m_3 p_4$	$s_6 a p_3 p_5 z$
49	$s_{12} a c_2 c_4 k w^2$	$s_7 a c_3 m_4 p_1 v$	$s_7 a c_6 m_4 m_5 p_2$	$s_7 a c_5 m_2 m_3 p_1 v$	$s_7 a c_6 m_3 p_4$	$s_6 a p_3 p_5 z$
50	$s_{12} a c_2 c_4 k v w^2$	$s_{12} a c_3 m_4 p_1 v w^2$	$s_7 a c_6 m_4 m_5 p_2 v^2$	$s_{12} a c_5 m_2 m_3 p_1 v w^2$	$s_7 a c_6 m_3 p_4 v$	$s_6 a p_3 p_5 z$
51	$s_{12} a c_2 c_4 k$	$s_{12} a c_3 n_4 p_1 v w$	$s_{12} a c_6 m_4 m_5 p_2 v^2 w^2$	$s_{12} a c_5 n_2 n_3 p_1 v^2 w$	$s_{12} a c_6 m_3 p_4 v w^2$	$s_6 a p_3 p_5 v^2 z$
52	$s_5 a c_2 c_4 k$	$s_{12} a c_3 n_4 p_1$	$s_{12} a c_6 n_4 n_5 p_2 v w$	$s_{12} a c_5 n_2 n_3 p_1$	$s_{12} a c_6 n_3 p_4 v w$	$s_{12} a p_3 p_5 v w^2 z$
53	$s_{13} a^2 c_2 c_4 w^2$	$s_7 a^2 c_3 n_4 p_1$	$s_{12} a c_6 n_4 n_5 p_2$	$s_7 a^2 c_5 n_2 n_3 p_1$	$s_{12} a c_6 n_3 p_4$	$s_{12} a p_3 p_5 w z$
54	$s_{13} a^4 c_2 c_4 w$	$s_{13} a^3 c_3 n_4 p_1 w^2$	$s_7 a^3 c_6 n_4 n_5 p_2$	$s_{13} a^3 c_5 n_2 n_3 p_1 w^2$	$s_7 a^2 c_6 n_3 p_4$	$s_{12} a p_3 p_5 z$
55	$s_{13} a^4 c_2 c_4$	$s_{13} a^4 c_3 n_4 p_1 w$	$s_{13} a^4 c_6 n_4 n_5 p_2 w^2$	$s_{13} a^5 c_5 n_2 n_3 p_1 w$	$s_{13} a^3 c_6 n_3 p_4 w^2$	$s_6 a^3 p_3 p_5 z$
56	$s_0 c_2 c_4$	$s_{13} a^4 c_3 n_4 p_1$	$s_{13} a^5 c_6 n_4 n_5 p_2 w$	$s_{13} a^5 c_5 n_2 n_3 p_1$	$s_{13} a^4 c_6 n_3 p_4 w$	$s_{13} a^3 p_3 p_5 w^2$
57	$s_0 c_2 c_4$	$s_0 c_3 p_1$	$s_0 c_6 p_2$	$s_0 c_5 p_1$	$s_{13} a^4 c_6 n_3 p_4$	$s_{13} a^3 p_3 p_5 w$
58	$s_0 c_2 c_4$	$s_0 c_3 p_1$	$s_0 c_6 p_2$	$s_0 c_5 p_1$	$s_0 c_6 p_4$	$s_{13} a^3 p_3 p_5$
59	$s_0 c_2 c_4$	$s_0 c_3 p_1$	$s_0 c_6 p_2$	$s_0 c_5 p_1$	$s_0 c_6 p_4$	$s_0 p_3 p_5$